On the physics of propagating Bessel modes in cylindrical waveguides

J. E. Gómez-Correa[a)]
Facultad de Ingeniería Mecánica y Eléctrica, Universidad Autónoma de Nuevo León, C.P. 66451 Nuevo León, Mexico and Centro de Investigación Científica y de Educación Superior de Ensenada, Unidad Monterrey, C.P. 66629 Nuevo León, Mexico

S. E. Balderas-Mata[b)]
Departamento de Electrónica, Universidad de Guadalajara, C.P. 44840 Guadalajara, Jalisco, Mexico

V. Coello[c)]
Centro de Investigación Científica y de Educación Superior de Ensenada, Unidad Monterrey, C.P. 66629 Nuevo León, Mexico

N. P. Puente[d)]
Facultad de Ingeniería Mecánica y Eléctrica, Universidad Autónoma de Nuevo León, C.P. 66451 Nuevo León, Mexico

J. Rogel-Salazar[e)]
Science and Technology Research Institute, School of Physics Astronomy and Mathematics, University of Hertfordshire, Hatfield, Herts. AL10 9AB, United Kingdom and Blackett Laboratory, Department of Physics, Imperial College London, Prince Consort Road, London SW7 2BZ, United Kingdom

S. Chávez-Cerda[f)]
Departamento de Óptica, Instituto Nacional de Astrofísica, Óptica y Electrónica, Apdo. Postal 51/216, Puebla, Mexico and Centro de Investigaciones en Óptica, C.P. 37150 León, Gto., Mexico

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In this paper, we demonstrate that by using a mathematical physics approach—focusing attention on the physics and using mathematics as a tool—it is possible to visualize the formation of the transverse modes inside a cylindrical waveguide. The opposite (physical mathematics) approach looks at the mathematical problem and then tries to impose a physical interpretation. For cylindrical waveguides, the physical mathematics route leads to the Bessel differential equation, and it is argued that in the core of the waveguide there are only Bessel functions of the first kind in the description of the transverse modes. The Neumann functions are deemed non-physical due to their singularity at the origin and are eliminated from the final description of the solution. In this paper, by combining geometric optics and wave optics concepts, we show that the inclusion of the Neumann function is physically necessary to describe fully and properly the formation of the propagating transverse modes. With this approach, we also show that the field outside a dielectric waveguide arises in a natural way.
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I. INTRODUCTION

In the work of modeling a physical phenomenon, it is sometimes easy to concentrate on understanding the intricacies of the mathematical methods employed to obtain the solutions that attempt to describe the problem at hand, and to be distracted from the significance of the actual physical process in question. Once one has successfully found the mathematical solution, the physical process itself may not be properly addressed and emphasized, and in the worst cases it may even be lost.

In this article, we deal with the description of the modes in cylindrical waveguides, where typically the physical phenomena involved are treated in such a way that the balance between the mathematical methods used and the physical constraints of the problem is tipped towards the former, leading to a somewhat unsatisfactory physical description of the formation of the modes in the waveguides. In this regard, it is worth remembering the words of Sommerfeld in the preface of his book on partial differential equations,[1] a book with some physics, but whose subject was mathematics: "*We do not really deal with mathematical physics, but physical mathematics; not with the mathematical formulation of physical facts, but with the physical motivation of mathematical methods. The oft-mentioned prestabilized harmony between what is mathematically interesting and what is physically important is met at each step and lends an esthetic - I should like to say metaphysical - attraction to our subject*." Of course, the last statement can apply in either direction, mathematical physics or physical mathematics.

In order to provide a framework for the work presented here, let us exemplify the difference between the physical mathematics (p-mathematics) and the mathematical physics (m-physics) approaches alluded to by Sommerfeld and which we will use in the rest of this paper. Consider the problem of finding the transverse modes in a planar (or slab) dielectric waveguide. On the one hand, the p-mathematics approach is used when presenting the mathematical solutions and boundary conditions in different regions of the slab. These solutions and boundary conditions are matched at each interface between regions, and the description is said to be obtained. On the other hand, the m-physics approach will look instead at the physical wave as it travels through each of the regions of the slab. We can then observe how the wave is reflected and transmitted as it reaches an interface between the media in question and then set the equations of the mathematical model according to the boundary conditions. We would like to note that both approaches attempt to describe the same phenomenon, however, they may emphasize different aspects of the description.

In the case of the example used above, the p-mathematics approach will give the sine and cosine functions as solutions inside the waveguide, and the arguments of these functions are set to satisfy the boundary conditions. In a different manner, the m-physics approach provides a more extensive physical picture, e.g., that within the waveguide there will be traveling waves suffering reflections at both interfaces that in turn will have transverse components with the same frequency but traveling in opposite directions. In this way, for some particular conditions, the transverse modes in waveguides happen to be transverse standing waves also referred to as stationary waves.[2,3]

The m-physics approach is sometimes discussed in the study of planar waveguides, mostly for the cosine function solution[4–6] but, to the best of our knowledge, it has never been used for cylindrical waveguides and optical fibers. For the latter two, the p-mathematics approach is the standard one, presenting the solutions in terms of the Bessel functions of the first and second kind, then discarding the latter arguing that they are *unphysical* or not the proper ones since they are singular at the axis of the waveguide.[7–15] Unfortunately, for such arguments, based on the mathematical properties of the functions, without further consideration of the physical meaning of these properties, the physics of what really might be happening in the physical system of the cylindrical waveguides can be lost.

In this work, we endeavor to use the mathematical formulation of the physical facts, i.e., the *mathematical physics* approach, to present a detailed physical analysis of the formation of modes in cylindrical waveguides. We demonstrate physically, with the help of propagating wave analysis, that the well-known Bessel modes are in fact the result of the interference of the counter-propagating transverse components of traveling conical waves. Contrary to the prevailing approach in the literature, we show that to fully describe the propagating nature of the wave field in the core and the cladding, it is necessary to use both Hankel functions, constructed by the complex superposition of the Bessel function of the first kind J_m, and Bessel functions of the second kind, or Neumann functions, N_m. Moreover, the singular behavior of these functions, in particular, of the Neumann functions, at the origin, is easily explained in clear physical terms. Finally, our mathematical physics traveling-wave approach shows how the traveling wave described by the Hankel function within the waveguide becomes, in a natural way, an evanescent wave at the interface between the core and the cladding of dielectric cylindrical waveguides, in the same manner as in planar waveguides.

II. MATHEMATICAL PHYSICS OF CYLINDRICAL WAVEGUIDES

In the literature on electromagnetic theory of wave propagation in cylindrical waveguides, one encounters the transverse fields described as satisfying the Bessel differential equation. The set of independent solutions of this equation are the Bessel functions of the first kind, J_m, as well as the Bessel functions of the second kind, N_m, also known as Neumann functions.

In the literature related to cylindrical waveguides, the treatment usually follows what we are referring to as the p-mathematics approach. In other words, a treatment inclined more towards the mathematical approach. It is commonly argued that a general solution can be constructed as a linear combination of the two types of Bessel functions, namely, $E(\rho) = AJ_m(\rho) + BN_m(\rho)$, where ρ is the cylindrical radial coordinate and where A and B are constants.[7–15] This superposition is mathematically correct but unfortunately it may not give complete physical insight to the problem at hand. The argument used in this p-mathematics approach is along the lines of "looking for the physical solution" inside the cylindrical waveguide, which leads to the conclusion that the constant B in the superposition above needs to be zero; the Neumann function has to be discarded because, for any m, it is singular at the origin $\rho = 0$, diverging to minus infinity. This argument considers that such behavior of the solution is inconsistent with physical fields within the core of the waveguide and that the only "proper" and physically allowed solution that is bounded is J_m, the Bessel function of the first kind.

The main aim of this paper is to show that in a full description and analysis of a cylindrical waveguide, the Neumann functions become a natural part of the solution and, furthermore, that their presence provides a complete physical picture of how the modes are formed in cylindrical waveguides. In order to show this, we first solve the wave equation or the Helmholtz equation in cylindrical coordinates, $\nabla^2 E(\rho, \varphi, z) + k^2 E(\rho, \varphi, z) = 0$. By applying separation of variables and using $E(\rho, \varphi, z) = R(\rho)\Phi(\varphi)Z(z)$ as an ansatz, we get three differential equations. Those for $Z(z)$ and $\Phi(\varphi)$ are

$$\frac{d^2 Z(z)}{dz^2} + k_z^2 Z(z) = 0, \tag{1}$$

$$\frac{d^2 \Phi(\varphi)}{d\varphi^2} + m^2 \Phi(\varphi) = 0, \tag{2}$$

whose solutions (in complex form) are, respectively,

$$Z(z) = e^{\pm i k_z z} \tag{3}$$

and

$$\Phi(\varphi) = e^{\pm i m \varphi}. \tag{4}$$

The third differential equation, corresponding to the radial part $R(\rho)$ of the ansatz, is the Bessel differential equation

$$\frac{d^2}{d\rho^2} R(\rho) + \frac{1}{\rho}\frac{d}{d\rho} R(\rho) + \left(k_\rho^2 - \frac{m^2}{\rho^2}\right) R(\rho) = 0, \tag{5}$$

where we have defined $k_\rho^2 = k^2 - k_z^2$. The solutions in complex form are given by

$$H_m^{(1)}(k_\rho \rho) = J_m(k_\rho \rho) + iN_m(k_\rho \rho), \tag{6}$$

and

$$H_m^{(2)}(k_\rho\rho) = J_m(k_\rho\rho) - iN_m(k_\rho\rho), \tag{7}$$

which are known as the Hankel functions of the first and second kind, respectively. The Hankel functions are singular at $\rho = 0$ due to the presence of the singularity of the Neumann function. Below we will show, however, that this singularity has an actual physical meaning. Notice that Eq. (6), being the Green's function of the Helmholtz equation, represents radially outgoing cylindrical waves; it implies that we have a source of light that radiates energy radially.

We can now focus our attention on the physics of these solutions. To simplify the description and the visualization, let us take $m = 0$; the observations below apply as well for any m. With the z-dependence included, the solutions of the wave equation are $H_0^{(1)}(k_\rho\rho)e^{ikz}$ and $H_0^{(2)}(k_\rho\rho)e^{ikz}$. These functions represent conical waves with a total wave vector $\vec{k} = k_\rho\hat{\rho} + k_z\hat{z}$, as shown in Fig. 1(a). The zeroth-order Hankel function of the first kind $H_0^{(1)}$ describes radially symmetric cylindrical waves traveling away from the axis (outgoing waves). The radial component of the outgoing conical wave on reflection at the surface of the waveguide becomes an incoming conical wave described by $H_0^{(2)}$—the zeroth order Hankel function of the second kind $H_0^{(2)}$ represents waves traveling towards the axis (incoming waves). This situation is easy to visualize if the propagation of the conical wavefronts is known. The evolution of the propagation of a transverse section of the conical wavefronts is shown in Fig. 2, where the dots that are not black represent an outgoing conical wave while the black dots represent an incoming conical wave. It is possible to observe that a reflected incoming conical wave is generated when the outgoing one has reached the border and that the incoming conical wave is transformed into an outgoing wave when the former passes through the axis of the waveguide. It is important to note that the transverse section of the conical wavefronts is generated by the points ACDE from Fig. 1, and the transverse section of the wavefronts is divided into representative sections, solid circles in Fig. 2.

Because each wave is the complex conjugate of the other, the incoming and outgoing waves have the same frequency and amplitude and, consequently, their transverse radial components move in opposite directions; in superposition the singular Neumann functions cancel out. An alternative way to explain this fact is by means of mathematical physics: the cylindrical waveguide supports cylindrical incoming waves traveling towards the longitudinal axis ($\rho = 0$). As they get closer and closer to the axis, the waves "collapse" into a line, and it is actually this line that simultaneously acts as the source from which the outgoing cylindrical waves emanate. This is the physical explanation of the singularity of the Hankel functions: within the waveguide there is simultaneously a sink and a source that cancel out, resulting in the J_0 non-singular stationary wave solution.[16] In the core of the cylindrical waveguide, both waves exist simultaneously; that is, the solution must be given by

$$E(\rho, z) = \left[H_0^{(1)}(k_\rho\rho) + H_0^{(2)}(k_\rho\rho)\right]e^{ik_z z} = 2J_0(k_\rho\rho)e^{ik_z z}. \tag{8}$$

In order to show the propagating wave behavior described by the Hankel functions, we introduce their asymptotic approximations and incorporate the harmonic temporal dependence $e^{-i\omega t}$ of the wave equation[17]

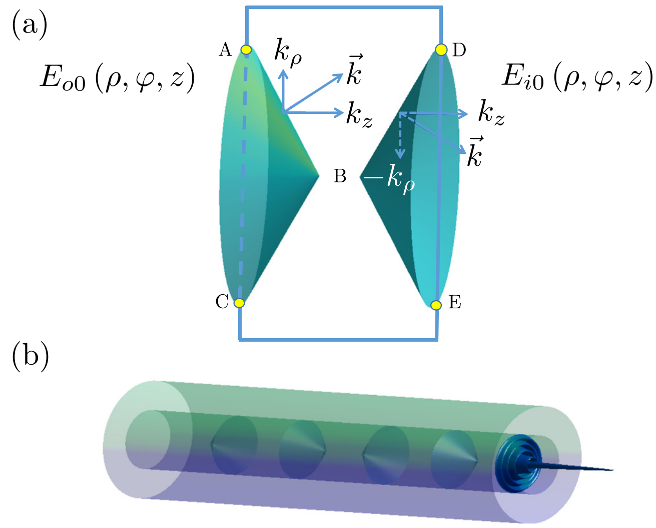

Fig. 1. (a) Representation of conical counter-propagating waves that are solutions to the Helmholtz equation in a cylindrical waveguide. Each conical wave corresponds to a Hankel function. (b) The sum of conical counter-propagating waves generate a Bessel function of order zero.

$$H_m^{(1,2)}(k_\rho\rho)e^{ik_z z} \approx \frac{A}{\sqrt{k_\rho\rho}}e^{-i(\omega t \mp k_\rho\rho + k_z z) - i\frac{\pi}{2}\left(m+\frac{1}{2}\right)}. \tag{9}$$

In this expression, the traveling wave behavior and the conical nature of the wavefronts are clear.

One may wonder how large the argument of the Hankel functions must be in order for the asymptotic expression to be valid. One finds in the literature (see, e.g., Ref. 17) that this expression can be used when $k_\rho\rho \gg (4m^2 - 1)/8$ with $m \geq 1$. To give a quantitative example, we can require that the error be of the order of 10^{-2} or less; to achieve this we must have

$$k_\rho\rho \geq 5\frac{|4m^2 - 1|}{8}. \tag{10}$$

The approximation would now have the required accuracy for approximating the Bessel functions of the first kind as well as for the Neumann functions, including those of zero order. A comparison is shown in Fig. 3 (for $m = 0$) where the Bessel and Neumann functions, together with their asymptotic approximations, are superposed in the top plot, and the simple differences are shown in bottom plot.

In a slab waveguide, there exist incident and reflected waves inside it, as described in Refs. 4 and 5. We note that inside a cylindrical waveguide there also exist incident as well as reflected waves at and from the cylindrical wall of the waveguide. In both cases, when the waves are added together they form stationary waves inside the waveguide.

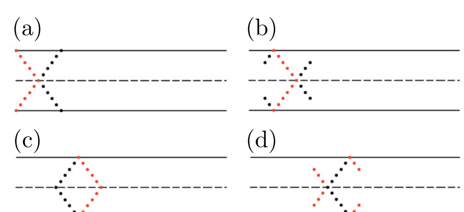

Fig. 2. The propagation evolution of the conical wavefronts.

For a slab waveguide, this is discussed in the literature; for a cylindrical waveguide we discuss this here for the first time. From Eq. (8), the solution can be defined as the sum of conical counter-propagating waves.

It is well accepted that the stationary modes inside a slab waveguide are generated by the sum of counter-propagating waves bouncing back and forth from the plane walls of the slab. With this slab-waveguide picture in mind, we introduce another approach to the description of the stationary waves inside a cylindrical waveguide. The latter is possible by rotating the slab waveguide π radians, as shown in Ref. 18. Since plane waves in cylindrical coordinates can be written as $e^{ik_\rho(x\cos\varphi+y\sin\varphi)+ik_z z}$, the continuous interference of these counter-propagating plane waves in the π rotation can be represented by

$$\frac{1}{\pi}\int_0^{\pi}\cos\left[k_\rho(x\cos\varphi+y\sin\varphi)\right]e^{ik_z z}d\varphi = \frac{1}{2\pi}\int_0^{2\pi}e^{ik_\rho(x\cos\varphi+y\sin\varphi+ik_z z)}d\varphi \equiv J_0(k_\rho\rho)e^{ik_z z}. \quad (11)$$

In a similar way, under a change of variables it can be demonstrated that if the plane waves have an azimuthal phase shift of the form $e^{im\varphi}$ as they rotate, the resulting field is described by

$$J_m(k_\rho\rho)e^{-im\theta+ik_z z} = \frac{1}{2\pi}\int_0^{2\pi}e^{ik_\rho\rho(\cos\alpha+\sin\alpha)-im\alpha+ik_z z}d\alpha\,, \quad (12)$$

where θ is now the azimuthal variable.

For $m=0$, the wavefront generated by the rotation of the plane waves forms two conical surfaces; in Fig. 1, cone ABC represents the outgoing conical wave incident to the inside wall of the waveguide, and cone DBE represents the reflected conical incoming wavefront. This situation is easy to visualize due to the change in sign of the k_ρ component. From Fig. 1, we can see that the vector k and its components k_ρ and k_z are coplanar vectors. Notice that the integral in Eq. (11) creates the cone of wavevectors.

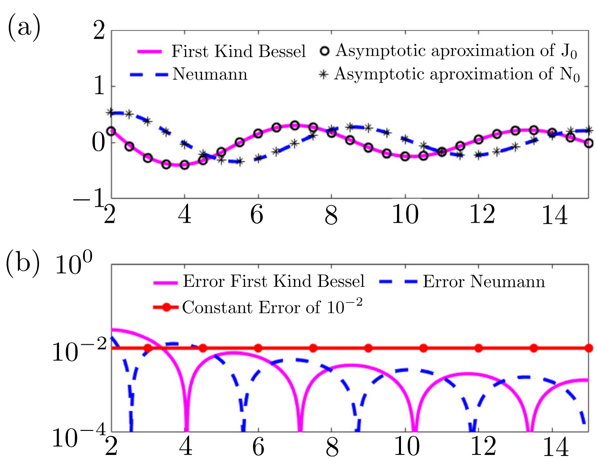

Fig. 3. (a) The continuous curve is the Bessel function of the first kind and the dashed curve is the Neumann function. The black asterisks and the open circles represent the asymptotic approximations for the Bessel function and the Neumann function, respectively. (b) The horizontal line represents an error of 10^{-2}. The continuous curve shows the difference between the Bessel function and its asymptotic approximation; the dashed curve shows the difference between the Neumann function and its asymptotic approximation.

Also observe that with the rotation of the planar slab description the singular Neumann function cannot be constructed, and this solution is discarded in a natural way. Physically, we can also deduce this from Eqs. (6) and (7); in order to get the Neumann solution we would have to subtract the waves, which would imply a relative phase of π between the incoming and outgoing conical waves along the whole waveguide, something that does not occur. In this manner, the approach presented here contributes to making the modes inside the cylindrical waveguide more physically understandable.

We will now demonstrate how the traveling waves inside the core become, in a natural way, the evanescent wave at the cladding. We have said that in the core the function $H_0^{(1)}(k_\rho\rho)e^{ikz}$ represents the outgoing wave. At the cylindrical waveguide surface $\rho=a$ the incident wave, by total reflection, must give rise to an evanescent field outside the core. In the same way as occurs with the slab waveguide, the transverse wave number becomes purely imaginary, $k_{\rho t}=i\kappa_{\rho t}$, resulting in the Hankel function of the first kind being transmitted to the outer part of the cylindrical waveguide. In this case, the solution in this region is given by

$$E(\rho>a,z) = H_0^{(1)}(i\kappa_\rho\rho)e^{ik_z z}. \quad (13)$$

This equation can be rewritten using the modified Bessel function of the second kind, $K_m(\kappa_\rho\rho)=\frac{\pi}{2}i^{m+1}H_m^{(1)}(i\kappa_\rho\rho)$,[17,19] and for $m=0$ we have

$$E(\rho>a,z) = \frac{2}{\pi}K_0(\kappa_\rho\rho)e^{i\left(k_z z+\frac{\pi}{2}\right)}. \quad (14)$$

This result demonstrates the natural way in which the outgoing traveling wave becomes an evanescent wave described by the modified Bessel function $K_0(\kappa_\rho\rho)$.

The transmitted wave right at the cladding is

$$J_0(\kappa_\rho a) = \left|\frac{e^{i\frac{\pi}{2}}}{\pi}K_0(\kappa_\rho a)\right|, \quad (15)$$

where we can observe that the amplitude coefficient of the transmitted wave is given by $|e^{i\frac{\pi}{2}}/\pi|$. Moreover, the $\pi/2$ phase shift can be interpreted as the rotation of the radial component of the wavevector to create the surface waves.

We remark that it was not necessary to make any mathematical assumptions in order to end up with the radially evanescent waves and the longitudinally traveling surface waves of Eq. (14) at the surface of the cylindrical waveguide. In particular, this is contrary to what is done in the physical mathematics approach where the modified Bessel function K_m is usually chosen over the modified Bessel function I_m because the latter function grows to infinity at large ρ, while the former decays. In our analysis, the direct use of the Hankel function as a solution inside the waveguide gives rise naturally to the evanescent wave in the form of the modified Bessel function K_m without having to make any further mathematical assumptions. Furthermore, in the core the solutions we have obtained are also able to describe the phenomena observed in tubular mirrors.[20,21]

Throughout this section, for the sake of clarity, the order of the Hankel functions has been taken as zero ($m=0$). In Sec. III, we will briefly describe the solution of the wavefronts of higher-order modes ($m\neq 0$) inside of the cylindrical

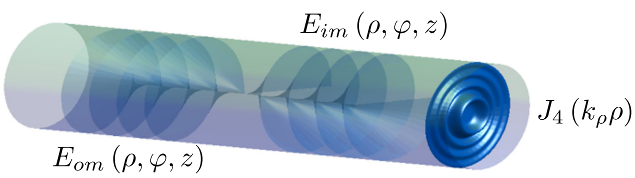

Fig. 4. Conical helicoidal wavefronts for a cylindrical waveguide high-order mode with $m = 4$.

waveguide. These modes are also known as skew modes in the optical fibers literature.

III. HIGHER-ORDER MODES IN CYLINDRICAL WAVEGUIDES

We will now turn our attention to the higher-order modes in cylindrical waveguides, which have the solutions

$$\begin{aligned} E_{om}(\rho,\varphi,z) &= e^{-i(k_z z+m\varphi)}H_m^{(1)}(k_\rho\rho) \\ E_{im}(\rho,\varphi,z) &= e^{-i(k_z z+m\varphi)}H_m^{(2)}(k_\rho\rho) \end{aligned} \tag{16}$$

in the core, and the solution

$$E(\rho,\varphi,z) = \frac{2}{\pi}e^{-i\left[k_z z+m\left(\varphi-\frac{\pi}{2}\right)-\frac{\pi}{2}\right]}K_m(\kappa_\rho\rho), \tag{17}$$

in the cladding. The factor $e^{-im\varphi}$ represents the phase rotating m times in a period of 2π, as shown in Fig. 4. This figure shows the conical helicoidal wavefronts that, for higher-order modes, take the place of the conical wavefronts corresponding to $m = 0$ as shown earlier.

As the wavefront is a conical helicoid, the propagation vector $\vec{k}$ follows a screw-like helical trajectory and its components no longer lie on a plane perpendicular to the tangential plane of the cylinder at any given point.

IV. CONCLUSIONS

We have presented a discussion of the differences between the physical mathematics and mathematical physics approaches used to describe problems in physics. We have demonstrated that in using the mathematical physics approach some physical aspects of the propagation of electromagnetic waves in cylindrical waveguides are recovered that would be lost using the physical mathematics approach.

Having the traveling wave idea in mind, we demonstrated that the waves inside the cylindrical waveguide are in fact propagating conical waves. From a physical mathematics point of view, these conical waves would never be seen because they are described by the Hankel functions of the first and second kinds, which have a singularity due to the presence of the singular Neumann functions. Nonetheless, with the aid of the mathematical physics picture, we have shown that the Neumann functions are a very important component in the full description of the physics of these conical waves. The standing waves inside the cylindrical waveguide, which have a profile given by the Bessel function of the first kind, are the result of the transverse counter-propagating component of the conical waves described with the Hankel functions. The counter-propagation results in the superposition of the incoming and outgoing waves canceling out the term with the Neumann functions in a natural manner, without the need to arbitrarily discard them. Also, we demonstrated how the outgoing wave becomes, in a straightforward manner, the evanescent field when the condition for total internal reflection is satisfied.

In general, the physics oriented method presented in this paper, the mathematical physics approach, gives more physically rich insights into the modes of a cylindrical waveguide than are given by the physical mathematics methods that are more prevalent in the literature. This picture can also be applied in quantum mechanics, elasticity, and other fields in physics where wave equations appear in the corresponding physical models.

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[a]Electronic mail: jesus.gomezcrr@uanl.edu.mx
[b]Electronic mail: sandra.balderas@cucei.udg.mx
[c]Electronic mail: vcoello@cicese.mx
[d]Electronic mail: norma.puenterm@uanl.edu.mx
[e]Electronic mail: j.rogel@physics.org
[f]Electronic mail: sabino@inaoep.mx

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